

A DOUBLE BOUNDED VERSION OF SCHUR'S PARTITION THEOREM

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Dedicated to the memory of Paul Erdős

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Schur's partition theorem states that the number of partitions of n into distinct parts $\equiv 1, 2 \pmod{3}$ equals the number of partitions of n into parts which differ by ≥ 3 , where the inequality is strict if a part is a multiple of 3. We establish a double bounded refined version of this theorem by imposing one bound on the parts $\equiv 0, 1 \pmod{3}$ and another on the parts $\equiv 2 \pmod{3}$, and by keeping track of the number of parts in each of the residue classes $\pmod{3}$. Despite the long history of Schur's theorem, our result is new, and extends earlier work of Andrews, Alladi-Gordon and Bressoud. We give combinatorial and q -theoretic proofs of our result. The special case $L = M$ leads to a representation of the generating function of the underlying partitions in terms of the q -trinomial coefficients extending a similar previous representation of Andrews.

1. Introduction

Schur's celebrated partition theorem of 1926 is the following result [11]:

Theorem S. *The number of partitions of n into distinct parts $\equiv 1, 2 \pmod{3}$ equals the number of partitions of n into parts differing by ≥ 3 , where consecutive multiples of 3 cannot occur as parts.*

Subsequently Gleissberg [9] extended [Theorem S](#) to an arbitrary modulus $m \geq 3$, and established a stronger correspondence involving the number of parts.

Several proofs of Schur's theorem by a variety of approaches are known, most notably by Andrews [4] using generating functions, by Bressoud [7]

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involving a combinatorial bijection, and by Andrews [5] using a representation involving the q -trinomial coefficients. Alladi and Gordon [3] obtained generalizations and refinements using a new technique called *the method of weighted words*. They viewed a strong refinement of Schur's theorem as emerging out of the *key identity*

$$(1.1) \quad \sum_{i,j \geq 0} A^i B^j \sum_{k=0}^{\min(i,j)} \frac{q^{T_{i+j-k}+T_k}}{(q)_{i-k}(q)_{j-k}(q)_k} = \sum_{i,j \geq 0} \frac{A^i B^j q^{T_i+T_j}}{(q)_i (q)_j} = (-Aq)_\infty (-Bq)_\infty$$

under the transformations

$$(1.2) \quad \begin{array}{ll} \text{(dilation)} & q \mapsto q^3, \\ \text{(translations)} & A \mapsto Aq^{-2}, \quad B \mapsto Bq^{-1}. \end{array}$$

In (1.1) and in what follows, $T_n = n(n+1)/2$ is the n -th triangular number, and the symbols $(A)_n$ are defined by

$$(1.3) \quad (A)_n = (A; q)_n = \begin{cases} \prod_{j=0}^{n-1} (1 - Aq^j), & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ \prod_{j=1}^{-n} (1 - Aq^{-j})^{-1}, & \text{if } n < 0 \end{cases}$$

when n is an integer, and

$$(1.4) \quad (A)_\infty = \lim_{n \rightarrow \infty} (A)_n = \prod_{j=0}^{\infty} (1 - Aq^j), \quad \text{for } |q| < 1.$$

Our goal here is to establish the finite identity (2.1) stated in Section 2, from which (1.1) follows when $L, M \rightarrow \infty$. When the transformations (1.2) are applied to (2.1), the combinatorial interpretation yields Theorem 3 in Section 4, which is the double bounded refined version of Theorem S, with different bounds on the parts $\equiv 0, 1 \pmod{3}$ and $\equiv 2 \pmod{3}$, and where the number of parts in the three residue classes can be specified. Although there is a long and rich history on Schur's theorem, and refinements keeping track of the number of parts in residue classes are known (such as what follows from the infinite identity (1.1)), our result with bounds on parts in residue classes never seems have been stated in the literature.

In Section 2 we give a proof of the finite identity (2.1) by making use of the q -Chu-Vandermonde summation. Then we give two more proofs of (2.1) with the conditions $L, M \geq i+j$ and $i, j \geq 0$, in which case the identity has combinatorial interpretation. More precisely, the second proof in Section 3 utilizes Durfee rectangles and extends the ideas in [3], and the third proof which is combinatorial and bijective (see Section 4) uses the methods in [3] and [7].

The case $L = M$ in (2.1) is discussed in Section 5. Alladi and Gordon [3] had shown that the refined Schur theorem underlying (1.1) emerged from the study of the numerator of a certain infinite continued fraction. In Section 5 we show that our generating functions corresponding to the bound L on the parts are the numerator convergents to this continued fraction. In addition this also leads to a representation involving the q -trinomial coefficients (see Section 5) with two free parameters A and B in the summation, extending a result of Andrews [5] who had previously obtained such a representation with one free parameter.

The method of weighted words was substantially improved by Alladi, Andrews, and Gordon [1], to obtain generalizations and refinements of a deep partition theorem of Göllnitz. They did this by proving the infinite key identity (6.1) which is considerably deeper than (1.1). We have recently obtained a double bounded finite version of this Göllnitz key identity. This is stated as identity (6.3) in Section 6 and a complete discussion of it may be found in [2].

2. A double bounded key identity

Let L, M, i, j be arbitrary integers. Then we have

$$(2.1) \quad \sum_{k=0}^{\min(i,j)} q^{(i-k)(j-k)} \begin{bmatrix} M-i-j+k \\ k \end{bmatrix} \begin{bmatrix} M-j \\ i-k \end{bmatrix} \begin{bmatrix} L-i \\ j-k \end{bmatrix} = \begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} M-j \\ i \end{bmatrix},$$

where the q -binomial coefficients $\begin{bmatrix} n+m \\ n \end{bmatrix}$ are defined by

$$(2.2) \quad \begin{bmatrix} n+m \\ n \end{bmatrix}_q = \begin{bmatrix} n+m \\ n \end{bmatrix} = \begin{cases} \frac{(q^{m+1})_n}{(q)_n}, & \text{if } n \geq 0, \\ 0, & \text{if } n < 0. \end{cases}$$

As in (2.2), when the base for the binomial coefficient is q , we will suppress it, but if the base is anything other than q , it will be displayed. When $m \geq 0$, (2.2) yields the standard definition of the q -binomial coefficient which is symmetric in m and n , namely,

$$(2.3) \quad \begin{bmatrix} n+m \\ m \end{bmatrix} = \begin{cases} \frac{(q)_{n+m}}{(q)_m(q)_n}, & \text{if } n \geq 0, \\ 0, & \text{if } n < 0. \end{cases}$$

Let us now show that (2.1) yields (1.1) when $L, M \rightarrow \infty$. From (2.2) it follows that as $L, M \rightarrow \infty$, identity (2.1) reduces to

$$(2.4) \quad \sum_{k=0}^{\min(i,j)} \frac{q^{(i-k)(j-k)}}{(q)_k(q)_{i-k}(q)_{j-k}} = \frac{1}{(q)_i(q)_j}.$$

Next it is easy to verify that

$$(2.5) \quad T_{i+j-k} + T_k = T_i + T_j + (i-k)(j-k).$$

Thus (2.4) and (2.5) yield

$$(2.6) \quad \sum_{k=0}^{\min(i,j)} \frac{q^{T_{i+j-k}+T_k}}{(q)_{i-k}(q)_{j-k}(q)_k} = \frac{q^{T_i+T_j}}{(q)_i(q)_j},$$

which is (1.1), by comparing the coefficients of $A^i B^j$ on both sides.

Now we give a proof of (2.1). To this end we use (I.10) and (I.20) in Gasper and Rahman [8] to rewrite the left hand side of (2.1) as

$$(2.7) \quad \lim_{l \rightarrow L} \frac{(q^{l-i-j+1})_j (q^{M-i-j+1})_i}{(q)_j (q)_i} q^{ij} \sum_{k \geq 0} \frac{(q^{-j})_k (q^{-i})_k q^k}{(q)_k (q^{l-i-j+1})_k},$$

where we used limit definitions to make sure that all objects in (2.7) are well defined. The sum in (2.7) can be evaluated by appeal to the q -Chu-Vandermonde summation formula (II.6 in [8]) as

$$(2.8) \quad \frac{(q^{l-j+1})_j q^{-ij}}{(q^{l-i-j+1})_j}.$$

Combining (2.7) and (2.8) we see that the left hand side of (2.1) is

$$\lim_{l \rightarrow L} \frac{(q^{M-i-j+1})_i (q^{l-j+1})_j}{(q)_i (q)_j} = \begin{bmatrix} M-j \\ i \end{bmatrix} \begin{bmatrix} L \\ j \end{bmatrix}$$

which is the right hand side of (2.1), thereby completing the proof.

3. Proof using Durfee rectangles

We now give another proof of (2.1) when $L, M \geq i+j$ and $i, j \geq 0$. In this case, we can use (2.3) to write (2.1) in the fully expanded form

$$(3.1) \quad \sum_{k=0}^{\min(i,j)} q^{(i-k)(j-k)} \frac{(q)_{M-i-j+k}}{(q)_k (q)_{M-i-j}} \cdot \frac{(q)_{M-j}}{(q)_{i-k} (q)_{M-i-j+k}} \cdot \frac{(q)_{L-i}}{(q)_{j-k} (q)_{L-i-j+k}} \\ = \frac{(q)_L}{(q)_j (q)_{L-j}} \cdot \frac{(q)_{M-j}}{(q)_i (q)_{M-j-i}}.$$

Note that all terms involving the parameter M disappear from (3.1) after cancellation! Thus in this case, (2.1) is equivalent to the identity

$$(3.2) \quad \sum_{k=0}^{\min(i,j)} q^{(i-k)(j-k)} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} L-i \\ j-k \end{bmatrix} = \begin{bmatrix} L \\ j \end{bmatrix}$$

which can be proved combinatorially using Durfee rectangles.

The term $\begin{bmatrix} L \\ j \end{bmatrix}$ on the right hand side of (3.2) is the generating function of partitions π into $\leq j$ parts with each part $\leq L-j$. In the Ferrers graph of each such partition π , there is a maximal Durfee rectangle whose row length minus column length is $i-j$. Let this rectangle have $j-k$ rows and $i-k$ columns. The number of nodes in this rectangle is $(i-k)(j-k)$ and this accounts for the term

$$q^{(i-k)(j-k)}$$

in (3.2). Next, the portion of the Ferrers graph below this rectangle is a partition into no more than k parts each $\leq i-k$. The generating function of such partitions is

$$\begin{bmatrix} i \\ k \end{bmatrix}$$

Finally the portion of the Ferrers graph to the right of this rectangle is a partition into $\leq j-k$ parts each $\leq L-i-j+k$. The generating function of such partitions is

$$\begin{bmatrix} L-i \\ j-k \end{bmatrix}$$

Thus

$$q^{(i-k)(j-k)} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} L-i \\ j-k \end{bmatrix}$$

is the generating function of all such Ferrers graphs π having a Durfee rectangle of size $(i-k)(j-k)$. The parameter k depends on the particular Ferrers graph given. We need to sum over k to account for all Ferrers graphs under discussion, and so this proves (3.2).

Remark. This argument using Durfee rectangles is the same as in Alladi-Gordon [3] except that they did not impose a bound $L-j$ on the size of the parts.

4. Combinatorial proof

We now give a combinatorial proof of (2.1) by following the method of Alladi-Gordon [3]. In order to do this we need to briefly describe the generalization of Schur's partition theorem established in [3].

Alladi and Gordon consider integers occurring in three colors, of which two are primary represented by a and b , and one is secondary, represented by ab . The integer 1 occurs only in the primary colors a and b , and the integers $n \geq 2$ occur in all three colors, a, b , and ab . By the symbols a_n, b_n, ab_n , we mean the integer n occurring in colors a, b , and ab , respectively. To discuss partitions into colored integers we need an ordering among the symbols and the one we choose is

$$(4.1) \quad a_1 < b_1 < ab_2 < a_2 < b_2 < ab_3 < a_3 < b_3 \dots$$

The reason for the choice of this ordering will be explained soon.

By a partition of n we mean a sum of symbols arranged in decreasing order according to (4.1) such that the sum of the subscripts (weights) is n . For example, $b_5 + b_5 + a_5 + (ab)_5 + a_4 + b_3 + (ab)_3$ is a partition of 30. By a Type-1 partition we mean a partition of the form $n_1 + n_2 + \dots + n_\nu$, where the n_i are symbols in (4.1) such that the gap between the subscripts of consecutive symbols n_i and n_{i+1} in this partition is ≥ 1 , but with strict inequality if

$$(4.2) \quad \left. \begin{array}{l} n_i \text{ is of color } ab \text{ or if} \\ n_i \text{ is of color } a \text{ and } n_{i+1} \text{ is of color } b \end{array} \right\}.$$

In [3] it is shown that

$$\frac{q^{T_{i+j-k}+T_k}}{(q)_{i-k}(q)_{j-k}(q)_k}$$

is the generating function of all Type-1 partitions involving exactly $i-k$ a -parts, $j-k$ b -parts, and k ab -parts. The term

$$\frac{q^{T_i+T_j}}{(q)_i(q)_j}$$

is clearly the generating function for all vector partitions (π_1, π_2) where π_1 has i distinct a -parts and π_2 has j distinct b -parts. In view of these explanations, the combinatorial version of (1.1) is the following result [3].

Theorem 1. *Let $V(n; i, j)$ denote the number of vector partitions (π_1, π_2) of n where π_1 has exactly i distinct a -parts and π_2 has exactly j distinct b -parts.*

Let $S(n; r, s, t)$ denote the number of Type-1 partitions of n having r a -parts, s b -parts, and t ab -parts.

Then

$$V(n; i, j) = \sum_{\substack{r+t=i \\ s+t=j}} S(n; r, s, t).$$

A strong refinement of Schur's theorem follows from Theorem 1 by using the transformations (1.2) which converts the product on the right in (1.1) to

$$\prod_{m=1}^{\infty} (1 + Aq^{3m-2})(1 + Bq^{3m-1}).$$

Under the same transformations, the symbols a_n, b_n, ab_n become

$$(4.3) \quad \left. \begin{aligned} a_n &\mapsto 3n - 2, & b_n &\mapsto 3n - 1, & \text{for } n \geq 1, \\ ab_n &\mapsto 3n - 3, & & & \text{for } n \geq 2. \end{aligned} \right\}$$

Thus under (4.3), the ordering (4.1) is just the natural ordering among the positive integers

$$1 < 2 < 3 < \dots,$$

which explains the reason for the choice of this ordering. Also under the transformations (4.3), the gap conditions (4.2) governing the Type-1 partitions become the Schur gap conditions, namely, the gap between consecutive parts is ≥ 3 , with strict inequality if a part is a multiple of 3. Thus Theorem 1 is a strong refinement of Schur's theorem in the undilated form.

In order to make the combinatorial proof of (2.1) as clear as possible we need to give the combinatorial proof of Theorem 1 in [3] here. Once that is done, we can go through the steps of that proof by imposing bounds L and M on certain parts and then (2.1) will fall out easily.

The combinatorial proof of Theorem 1 will be illustrated with the vector partition $(\pi_1; \pi_2)$, where

$$\pi_1 : a_6 + a_5 + a_3 + a_2 + a_1, \quad \pi_2 : b_9 + b_8 + b_6 + b_4 + b_2 + b_1$$

Here $i=5$ and $j=6$.

Suppose in general that π_1 has i parts and π_2 has j parts.

Step 1. Decompose π_2 into π_4 and π_5 , where π_4 has the parts of π_2 which are $\leq i$ and π_5 has the remaining parts:

$$\pi_4 : b_4 + b_2 + b_1 \qquad \pi_5 : b_9 + b_8 + b_6.$$

Step 2. Consider the conjugate of the Ferrers graph of π_4 and circle the bottom node of each column. Denote this graph by π_4^* . Construct a graph π_6 where the number of nodes in each row is the sum of the number of nodes in the corresponding rows of π_1 and π_4^* . The parts of π_6 ending in circled nodes are ab -parts. The rest are a -parts:

$$\pi_6 = \pi_1 + \pi_4^* : ab_9 + ab_7 + a_4 + ab_3 + a_1.$$

Conversely, given π_6 , the columns ending with the circled nodes can be extracted to form π_4^* , and what remains after the extraction will be π_1 .

Step 3. Write the parts of π_5 in a column in descending order and below them write the parts of π_6 in descending order.

Step 4. Subtract 0 from the bottom element, 1 from the next element above, 2 from the one above that, etc., and display the new values as well as the subtracted ones in two adjacent columns $C_1|C_2$. The elements of C_2 have no color, while those of C_1 retain the colors of the parts from which they were derived.

Step 5. Rearrange the elements of C_1 in decreasing order given by (4.1) to form a column C_1^R .

Step 6. Finally, add the corresponding elements of C_1^R and C_2 to get a partition π_3 counted by $S(n; r, s, t)$. The colors of the parts of π_3 are those of the elements of C_1^R from which they were derived.

Step 3	Step 4		Step 5		Step 6
π_5/π_6	C_1	C_2	C_1^R	C_2	π_3
b_9	b_2	7	ab_5	7	ab_{12}
b_8	b_2	6	ab_4	6	ab_{10}
b_6	b_1	5	b_2	5	b_7
ab_9	ab_5	4	b_2	4	b_6
ab_7	ab_4	3	a_2	3	a_5
a_4	a_2	2	ab_2	2	ab_4
ab_3	ab_2	1	b_1	1	b_2
a_1	a_1	0	a_1	0	a_1

Each of these steps is a one-to-one correspondence, and so this is a bijective proof of [Theorem 1](#).

We now give a bijective proof of [\(2.1\)](#) when $L, M \geq i + j$ and $i, j \geq 0$ by discussing the above steps in reverse. For this purpose we need to consider the quantities $\nu(\pi; M) = \nu(M)$ and $\nu(\pi; L) = \nu(L)$, which are defined as follows.

1) If $L \geq M$ then $\nu(L) = 0$. If $L < M$ then $\nu(L)$ is the non-negative number such that there are exactly $\nu(L)$ parts in the interval $[L - \nu(L) + 2, M]$, all b parts $\leq L - \nu(L)$ and no part $= L - \nu(L) + 1$.

2) If $M \geq L$ then $\nu(M) = 0$. If $M < L$ then $\nu(M)$ is the non-negative number such that there are exactly $\nu(M)$ parts in the interval $[M - \nu(M) + 2, L]$, all ab , a parts $\leq M - \nu(M)$ and no part $= M - \nu(M) + 1$.

First using [\(2.5\)](#) we rewrite [\(2.1\)](#) in the equivalent form

$$\begin{aligned} & \sum_{k=0}^{\min(i,j)} q^{T_{i+j-k}+T_k} \begin{bmatrix} M-i-j+k \\ k \end{bmatrix} \begin{bmatrix} M-j \\ i-k \end{bmatrix} \begin{bmatrix} L-i \\ j-k \end{bmatrix} \\ (4.4) \quad & = q^{T_i+T_j} \begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} M-j \\ i \end{bmatrix}. \end{aligned}$$

Identity [\(4.4\)](#) can be proved combinatorially using the above steps as we show now.

First, we interpret

$$q^{T_k} \begin{bmatrix} M-i-j+k \\ k \end{bmatrix} \begin{bmatrix} M-j \\ i-k \end{bmatrix} \begin{bmatrix} L-i \\ j-k \end{bmatrix}$$

as the generating function for partitions having

$$\left. \begin{aligned} & \leq i-k \quad a\text{-parts, each} \leq M-j-i+k, \\ & \leq j-k \quad b\text{-parts, each} \leq L-i-j+k, \\ & \text{and } k \text{ distinct } ab\text{-parts with } ab_1 \text{ permitted, each} \leq M-i-j+k. \end{aligned} \right\}$$

Next, to this partition, we add 1 to the smallest part, 2 to the second smallest part, \dots , $i+j-k$ to the largest part, and retain the colors of the parts we started with. We then get

$$q^{T_{i+j-k}+T_k} \begin{bmatrix} M-j-i+k \\ k \end{bmatrix} \begin{bmatrix} M-j \\ i-k \end{bmatrix} \begin{bmatrix} L-i \\ j-k \end{bmatrix}$$

as the generating function for Type-1 partitions having

$$\left. \begin{aligned} & i-k \quad a\text{-parts} \leq M-\nu(M), \\ & j-k \quad b\text{-parts} \leq L-\nu(L), \\ & k \quad ab\text{-parts} \leq M-\nu(M). \end{aligned} \right\}$$

This is the interpretation of the summand on the left in (4.4) and the partition corresponds to one of the form π_3 in Step 6.

Going from Step 6 to Step 5, we subtract $0, 1, 2, 3, \dots, i+j-k-1$ in succession. Thus the partitions in column C_1^R would have:

$$\left. \begin{array}{l} i-k \text{ } a\text{-parts each} \leq M-i-j+k+1, \\ j-k \text{ } b\text{-parts each} \leq L-i-j+k+1, \\ k \text{ } \text{distinct } ab\text{-parts each} \leq M-i-j+k+1. \end{array} \right\}$$

Proceeding from Step 5 to Step 3, we do a rearrangement, and then we add $0, 1, 2, \dots, i-1$ to the a -parts and ab -parts, whereas we add $i, i+1, \dots, i+j-k-1$ to the b -parts. Thus π_5/π_6 in Step 3 is a partition with

$$\left. \begin{array}{l} i-k \text{ } \text{distinct } a\text{-parts each} \leq M-j+k, \\ j-k \text{ } \text{distinct } b\text{-parts in the interval } [i+1, L], \\ k \text{ } ab\text{-parts differing by } \geq 2 \text{ and all } \leq M-j+k. \end{array} \right\}$$

Finally in Step 2 the k ab -parts decompose into k a -parts and k b -parts, the latter being $\leq i$. In this process all a -parts are bounded by $M-j$. Thus we end up with partitions having

$$(4.5) \quad \left. \begin{array}{l} i \text{ } \text{distinct } a\text{-parts each} \leq M-j, \\ \text{and } j \text{ } \text{distinct } b\text{-parts each} \leq L. \end{array} \right\}$$

The generating function of the partitions satisfying (4.5) is

$$q^{T_i+T_j} \begin{bmatrix} M-j \\ i \end{bmatrix} \begin{bmatrix} L \\ j \end{bmatrix}$$

which is the right hand side of (4.4), thereby completing the combinatorial proof.

In view of the above proof and interpretation, we can improve Theorem 1 to the following double bounded form:

Theorem 2. *Let L, M, i, j be non-negative integers with $M \geq L \geq i+j$.*

Let $V(n; i, j, L, M)$ denote the number of vector partitions $(\pi_1; \pi_2)$ of n having i distinct a -parts each $\leq M-j$, and j distinct b -parts each $\leq L$.

Let $S(n; r, s, t, l, L, M)$ denote the number of Type-1 partitions of n having r a -parts $\leq M$, s b -parts $\leq L-l$, t ab -parts $\leq M$, $l = \nu(L)$ a, ab -parts which are $\geq L-l+2$, and no part $= L-l+1$.

Then

$$V(n; i, j, L, M) = \sum_{\substack{r+t=i \\ s+t=j}} \sum_l S(n; r, s, t, l, L, M).$$

NOTE: Since $M \geq L$ in [Theorem 2](#), we have $\nu(M) = 0$ and so the inner summation is over $l = \nu(L)$. If $L \geq M$, then $\nu(L) = 0$ and so there would be a similar theorem with the inner summation over $m = \nu(M)$.

Under the transformations (1.2), [Theorem 2](#) yields the following double bounded strong refinement of Schur's theorem which is new:

Theorem 3. *Let L, M, i, j , be non-negative integers with $M \geq L \geq i + j$.*

Let $P(n; i, j, L, M)$ denote the number of partitions of n into i distinct parts $\equiv 1 \pmod{3}$ each $\leq 3(M - j) - 2$, and j distinct parts $\equiv 2 \pmod{3}$ each $\leq 3L - 1$.

Let $G(n; r, s, t, l, L, M)$ denote the number of partitions of n into parts differing by ≥ 3 , where the inequality is strict if a part is a multiple of 3, and such that there are r parts $\equiv 1 \pmod{3}$ each $\leq 3M - 2$, s parts $\equiv 2 \pmod{3}$ each $\leq 3(L - l) - 1$, t parts $\equiv 0 \pmod{3}$ each $\leq 3M - 3$, l parts which are $> 3(L - l) + 2$, and no part $= 3(L - l)$ or $3(L - l) + 1$.

Then

$$P(n; i, j, L, M) = \sum_{\substack{r+t=i \\ s+t=j}} \sum_l G(n; r, s, t, l, L, M).$$

As in the case of [Theorem 2](#), there is a version of [Theorem 3](#) when $L > M \geq i + j$.

Remarks. There are other ways in which one might obtain double bounded versions of the identity (2.6). For instance, consider

$$(4.6) \quad (-Aq)_M (-Bq)_L = \sum_{i=0}^M \sum_{j=0}^L A^i B^j q^{T_i+T_j} \begin{bmatrix} M \\ i \end{bmatrix} \begin{bmatrix} L \\ j \end{bmatrix}.$$

We may use the q -binomial theorem to expand $(-Aq)_M$ and rewrite the left hand side of (4.6) as

$$(4.7) \quad (-Aq)_M (-Bq)_L = \sum_{i=0}^M A^i q^{T_i} (-Bq)_i (-Bq^{i+1})_{L-i} \begin{bmatrix} M \\ i \end{bmatrix}.$$

By expanding the factors $(-Bq)_i$ and $(-Bq^{i+1})_{L-i}$ using the q -binomial theorem once again, and rearranging, we get on comparison with (4.6) the following finite identity for Schur's theorem:

$$(4.8) \quad q^{T_i+T_j} \begin{bmatrix} M \\ i \end{bmatrix} \begin{bmatrix} L \\ j \end{bmatrix} = \sum_k q^{T_{i+j-k}+T_k} \begin{bmatrix} M \\ M-i, i-k, k \end{bmatrix} \begin{bmatrix} L-i \\ j-k \end{bmatrix},$$

where

$$\left[\begin{matrix} M \\ i, j, M-i-j \end{matrix} \right] = \frac{(q)_M}{(q)_i(q)_j(q)_{M-i-j}}$$

is the q -multinomial coefficient of order 3. Identity (4.8) has the advantage that the bounds on the parts enumerated by the left hand side are simple, namely, that the a -parts are bounded by M and the b -parts by L . But then the bounds on the parts of the partitions enumerated by the right hand are more complicated. The decomposition considered in (4.7) corresponds precisely to the decomposition of the partition π_2 into $\pi_4 + \pi_5$ in Step 1 above.

5. Other Connections

In this section we discuss the case $L=M$ of the double bounded key identity (4.4), namely, the case where all parts of $S(n; r, s, k)$ are bounded by b_L . In this case the product of the q -binomial coefficients on the right hand side of (4.4) becomes

$$(5.1) \quad \left[\begin{matrix} L \\ j \end{matrix} \right] \left[\begin{matrix} L-j \\ i \end{matrix} \right] = \frac{(q)_L}{(q)_j(q)_{L-j}} \frac{(q)_{L-j}}{(q)_i(q)_{L-i-j}} = \left[\begin{matrix} L \\ i, j, L-i-j \end{matrix} \right],$$

the q -multinomial coefficient (of order 3). Thus when $L=M$ (4.4) reduces to

$$(5.2) \quad \sum_{k=0}^{\min(i,j)} q^{T_{i+j-k}+T_k} \left[\begin{matrix} L-i-j+k \\ k \end{matrix} \right] \left[\begin{matrix} L-j \\ i-k \end{matrix} \right] \left[\begin{matrix} L-i \\ j-k \end{matrix} \right] \\ = q^{T_i+T_j} \left[\begin{matrix} L \\ i, j, L-i-j \end{matrix} \right].$$

Now multiply both sides of (5.2) by $A^i B^j$ and sum over i and j to get

$$(5.3) \quad \sum_{i,j \geq 0} A^i B^j \sum_{k=0}^{\min(i,j)} q^{T_{i+j-k}+T_k} \left[\begin{matrix} L-i-j+k \\ k \end{matrix} \right] \left[\begin{matrix} L-j \\ i-k \end{matrix} \right] \left[\begin{matrix} L-i \\ j-k \end{matrix} \right] \\ = \sum_{i,j \geq 0} A^i B^j q^{T_i+T_j} \left[\begin{matrix} L \\ i, j, L-i-j \end{matrix} \right].$$

Even though (5.3) is an immediate consequence of (4.4), it is instructive to give an independent proof of (5.3). We will do so by interpreting the left

hand side of (5.3) combinatorially, and by utilizing a not so well known recurrence formula for the q -multinomial coefficients for the right hand side (see (5.8) below). This leads to a connection with a continued fraction expansion considered by Alladi and Gordon [3] and extends a representation due to Andrews [5] involving the q -trinomial coefficients.

Denote the left hand side of (5.3) by $G_L(A, B; q)$. From the discussion in Section 4 it follows that $G_L(A, B; q)$ is the generating function of Type-1 partitions π where all parts are $\leq b_L$. That is

$$(5.4) \quad G_L(A, B; q) = \sum_{\substack{\pi \text{ of type 1} \\ \lambda(\pi) \leq b_L}} A^{\nu_a(\pi)} B^{\nu_b(\pi)} (AB)^{\nu_{ab}(\pi)} q^{\sigma(\pi)},$$

where $\sigma(\pi)$ is the sum of the parts of π , $\lambda(\pi)$ the largest part of π , and $\nu_a(\pi)$, $\nu_b(\pi)$, $\nu_{ab}(\pi)$, denote the number of a -parts, b -parts, and ab -parts of π respectively.

In Alladi and Gordon [3] it is shown that the generating function $G_L(A, B; q)$ satisfies the recurrence

$$(5.5) \quad G_L(A, B; q) = (1 + Aq^L + Bq^L)G_{L-1}(A, B; q) + ABq^L(1 - q^{L-1})G_{L-2}(A, B; q),$$

but they did not have the representation for $G_L(A, B; q)$ as the left hand side of (5.3).

Next, let $R_L(A, B; q)$ denote the right hand side of (5.3). We claim that R_L satisfies the same recurrence, namely,

$$(5.6) \quad R_L(A, B; q) = (1 + Aq^L + Bq^L)R_{L-1}(A, B; q) + ABq^L(1 - q^{L-1})R_{L-2}(A, B; q).$$

By considering the coefficient of $A^i B^j$ in R_L , we see that proving (5.6) is equivalent to showing

$$(5.7) \quad \begin{aligned} q^{T_i+T_j} \begin{bmatrix} L \\ i, j, L-i-j \end{bmatrix} &= q^{T_i+T_j} \begin{bmatrix} L-1 \\ i, j, L-i-j-1 \end{bmatrix} \\ &+ q^{T_{i-1}+T_j+L} \begin{bmatrix} L-1 \\ i-1, j, L-i-j \end{bmatrix} \\ &+ q^{T_i+T_{j-1}+L} \begin{bmatrix} L-1 \\ i, j-1, L-i-j \end{bmatrix} \\ &+ q^{T_{i-1}+T_{j-1}+L} (1 - q^{L-1}) \begin{bmatrix} L-2 \\ i-1, j-1, L-i-j \end{bmatrix}. \end{aligned}$$

By cancelling $q^{T_i+T_j}$ on both sides of (5.7), we get the following equivalent second order (in L) recurrence relation for the q -multinomial coefficients:

$$\begin{aligned}
 \left[\begin{matrix} L \\ i, j, L-i-j \end{matrix} \right] &= \left[\begin{matrix} L-1 \\ i, j, L-1-i-j \end{matrix} \right] + q^{L-i} \left[\begin{matrix} L-1 \\ i-1, j, L-i-j \end{matrix} \right] \\
 &\quad + q^{L-j} \left[\begin{matrix} L-1 \\ i, j-1, L-i-j \end{matrix} \right] \\
 (5.8) \quad &\quad + q^{L-i-j}(1-q^{L-1}) \left[\begin{matrix} L-2 \\ i-1, j-1, L-i-j \end{matrix} \right].
 \end{aligned}$$

This recurrence is symmetric in i and j but is not so well known; it can be derived from one of the six (non-symmetric) standard recurrences for the q -multinomial coefficients, namely,

$$\begin{aligned}
 \left[\begin{matrix} L \\ i, j, L-i-j \end{matrix} \right] &= \left[\begin{matrix} L-1 \\ i, j, L-1-i-j \end{matrix} \right] + q^{L-i-j} \left[\begin{matrix} L-1 \\ i, j-1, L-i-j \end{matrix} \right] \\
 (5.9) \quad &\quad + q^{L-i} \left[\begin{matrix} L-1 \\ i-1, j, L-i-j \end{matrix} \right]
 \end{aligned}$$

Indeed (5.9) implies that (5.8) is equivalent to

$$\begin{aligned}
 q^{L-i-j} \left[\begin{matrix} L-1 \\ i, j-1, L-i-j \end{matrix} \right] &= q^{L-j} \left[\begin{matrix} L-1 \\ i, j-1, L-i-j \end{matrix} \right] \\
 (5.10) \quad &\quad + q^{L-i-j}(1-q^{L-1}) \left[\begin{matrix} L-2 \\ i-1, j-1, L-i-j \end{matrix} \right]
 \end{aligned}$$

We may cancel the common factor q^{L-i-j} and $(q)_{L-1}$ in the numerator of (5.10) and the common factors $(q)_{j-1}$ and $(q)_{L-i-j}$ in the denominator and reduce (5.10) to

$$\frac{1}{(q)_i} = \frac{q^i}{(q)_i} + \frac{1}{(q)_{i-1}}$$

which is clearly true. Thus (5.8) is established, and consequently (5.6). Finally, since G_L and R_L satisfy the same initial conditions, (5.3) is proven.

Alladi and Gordon [3] viewed the left hand side of (1.1) as the numerator of

$$(5.11) \quad \left(1 + (A+B)q + \frac{ABq^2(1-q)}{1 + (A+B)q^2 + \frac{ABq^3(1-q^2)}{1+(A+B)q^3+\dots}} \right).$$

Thus the infinite key identity is the statement that the numerator of this continued fraction equals

$$\prod_{m=1}^{\infty} (1 + Aq^m)(1 + Bq^m)$$

Now let $P_L(A, B; q)$ denote the numerator of the L -th convergent to this continued fraction. Clearly P_L satisfies the recurrence

$$(5.12) \quad P_L(A, B; q) = (1 + Aq^L + Bq^L)P_{L-1}(A, B; q) + ABq^L(1 - q^{L-1})P_{L-2}(A, B; q).$$

By comparing (5.12) with (5.5) and checking initial conditions it follows that the numerator convergents P_L are precisely the generating functions G_L for Type-1 partitions π with $\lambda(\pi) \leq b_L$. The left hand side of (5.3) is the representation for the convergent P_L . When $L \rightarrow \infty$, this representation yields the left hand side of (1.1).

We conclude this section by producing a representation for G_L in terms of the q -trinomial coefficients, thereby extending a previous similar representation by Andrews [5]. To this end, use the transformations (1.2) to recast (5.3) in the form

$$(5.13) \quad \sum_{i,j} \sum_{k=0}^{\min(i,j)} A^i B^j q^{3(T_i+j-k+T_k)-2i-j} \begin{bmatrix} L-i-j+k \\ k \end{bmatrix}_{q^3} \begin{bmatrix} L-i \\ j-k \end{bmatrix}_{q^3} \begin{bmatrix} L-j \\ i-k \end{bmatrix}_{q^3} \\ = \sum_{i,j} A^i B^j q^{3(T_i+T_j)-2i-j} \begin{bmatrix} L \\ i, j, L-i-j \end{bmatrix}_{q^3}.$$

Next, replace i by $j + \tau$ on the right hand side of (5.13) to rewrite it as

$$(5.14) \quad \sum_{j,\tau} (Aq)^{j+\tau} (Bq^2)^j q^{3\binom{j+\tau}{2} + 3\binom{j}{2}} \begin{bmatrix} L \\ j+\tau, j, L-2j-\tau \end{bmatrix}_{q^3} \\ = \sum_{j,\tau} (AB)^j A^\tau q^{\tau\left(\frac{3\tau-1}{2}\right) + 3j(j+\tau)} \begin{bmatrix} L \\ j+\tau, j, L-2j-\tau \end{bmatrix}_{q^3}.$$

Following Andrews and Baxter [6] we define the generalized q -trinomial coefficients $\begin{pmatrix} L; q \\ \tau \end{pmatrix}_c$ by

$$(5.15) \quad \begin{pmatrix} L; q \\ \tau \end{pmatrix}_c = \sum_j c^j q^{j(j+\tau)} \begin{bmatrix} L \\ j+\tau, j, L-2j-\tau \end{bmatrix}_{q^3}.$$

Then from (5.3), (5.4), (5.13), (5.14) and (5.15), we get the representation

$$(5.16) \quad G_{3L-1}(Aq^{-2}, Bq^{-1}; q^3) = \sum_{\tau} A^{\tau} q^{\tau(\frac{3\tau-1}{2})} \left(\begin{matrix} L; q^3 \\ \tau \end{matrix} \right)_{c=AB}$$

for the generating function of all partitions enumerated by $B(n)$ having parts $\leq 3L - 1$. Previously Andrews [5] had utilized the q -trinomial coefficients to get the representation (5.16) where $c = AB = 1$ which corresponds to Gleissberg's refinement [9] of Schur's partition theorem.

6. A double bounded Göllnitz identity

One of the deepest results in the theory of partitions is a theorem of Göllnitz [10]:

Theorem G. *Let $P(n)$ denote the number of partitions of n into distinct parts $\equiv 2, 4$ or $5 \pmod{6}$.*

Let $G(n)$ denote the number of partitions of n into parts $\neq 1$, or 3 , such that the difference between the parts is ≥ 6 with strict inequality if a part is $\equiv 0, 1$ or $3 \pmod{6}$.

Then

$$G(n) = P(n).$$

The proof by Göllnitz [10] is very involved.

Alladi, Andrews, and Gordon [1], obtained substantial generalizations and refinements of Theorem G, and viewed this theorem as emerging out of the *key identity*

$$(6.1) \quad \sum_{i,j,k} A^i B^j C^k \sum_{\substack{i=\alpha+\delta+\varepsilon \\ j=\beta+\delta+\phi \\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_s+T_\delta+T_\varepsilon+T_\phi-1}(1-q^\alpha+q^{\alpha+\phi})}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi} \\ = (-Aq)_\infty (-Bq)_\infty (-Cq)_\infty$$

under the transformations

$$(6.2) \quad \left. \begin{array}{l} \text{(dilation)} q \rightarrow q^6, \\ \text{(translations)} A \rightarrow Aq^{-4}, B \rightarrow Bq^{-2}, C \rightarrow Cq^{-1}. \end{array} \right\}$$

Here $s = \alpha + \beta + \gamma + \delta + \varepsilon + \phi$. Note that the key identity (1.1) for Schur's theorem is the special case of (6.1) with $C = 0$.

We have recently obtained the following double bounded version of (6.1):

If i, j, k, L, M are given integers, then

$$\begin{aligned}
 & \sum q^{T_s+T_\delta+T_\varepsilon+T_{\phi-1}} \\
 & \left\{ q^\phi \begin{bmatrix} L-s+\alpha \\ \alpha \end{bmatrix} \begin{bmatrix} L-s+\beta \\ \beta \end{bmatrix} \begin{bmatrix} M-s+\gamma \\ \gamma \end{bmatrix} \begin{bmatrix} L-s \\ \delta \end{bmatrix} \begin{bmatrix} M-s \\ \varepsilon \end{bmatrix} \begin{bmatrix} M-s \\ \phi \end{bmatrix} \right. \\
 & + \begin{bmatrix} L-s+\alpha-1 \\ \alpha-1 \end{bmatrix} \begin{bmatrix} L-s+\beta \\ \beta \end{bmatrix} \begin{bmatrix} M-s+\gamma \\ \gamma \end{bmatrix} \\
 & \quad \left. \begin{bmatrix} L-s \\ \delta \end{bmatrix} \begin{bmatrix} M-s \\ \varepsilon \end{bmatrix} \begin{bmatrix} M-s \\ \phi-1 \end{bmatrix} \right\} \\
 & = \sum_{\tau \geq 0} q^{\tau(M+2)-T_\tau+T_{i-\tau}+T_{j-\tau}+T_{k-\tau}} \begin{bmatrix} L-\tau \\ \tau \end{bmatrix} \begin{bmatrix} L-2\tau \\ i-\tau \end{bmatrix} \\
 & \quad \begin{bmatrix} L-i-\tau \\ j-\tau \end{bmatrix} \begin{bmatrix} M-i-j \\ k-\tau \end{bmatrix}, \tag{6.3}
 \end{aligned}$$

where s is as in (6.1) and the summation in (6.3) is over $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$ satisfying the same conditions with respect to i, j, k as in (6.1). In (6.3) if we set either $i=0$ or $j=0$, then we get an identity equivalent to (4.4). If we set $k=0$, then (6.3) reduces to (5.3) which is (4.4) with $L=M$. By letting the parameters $L, M \rightarrow \infty$, only the term corresponding to $\tau=0$ on the right in (6.3) survives, and (6.3) reduces to

$$\sum_{\substack{i=\alpha+\delta+\varepsilon \\ j=\beta+\delta+\phi \\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_s+T_\delta+T_\varepsilon+T_{\phi-1}}(1-q^\alpha+q^{\alpha+\phi})}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi} = \frac{q^{T_i+T_j+T_k}}{(q)_i(q)_j(q)_k}$$

from which (6.1) follows if we multiply both sides by $A^i B^j C^k$ and sum over i, j, k . The proof of the new identity (6.3) is given in [2].

Finally, if $L=M$, then the sum on the right in (6.3) can be evaluated using the q -Pfaff-Saalschütz summation (see [8], formula II.12) in terms of three q -binomial coefficients where i, j, k occur cyclically

$$q^{T_i+T_j+T_k} \begin{bmatrix} L-k \\ i \end{bmatrix} \begin{bmatrix} L-i \\ j \end{bmatrix} \begin{bmatrix} L-j \\ k \end{bmatrix}.$$

In this case (6.3) can be interpreted in terms of partitions (see [2]).

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References

- [1] K. Alladi and G. E. Andrews, and B. Gordon: Generalizations and refinements of a partition theorem of Göllnitz, *J. reine angew. Math.*, **460** (1995), 165–188.
- [2] K. Alladi and A. Berkovich: A double bounded key identity for Göllnitz's (Big) partition theorem, in: Symbolic Computation, Number Theory, Special Functions, Physics, and Combinatorics (Frank Garaban and Mourad Ismail, Eds.), Developments in Mathematics, Vol. 4. Kluwer Acad. Publ. Dordrecht (2001), 13–32.
- [3] K. Alladi and B. Gordon: Generalizations of Schur's partition theorem, *Manus. Math.*, **79** (1993), 113–126.
- [4] G. E. Andrews: On Schur's second partition theorem, *Glasgow Math. J.*, **9** (1967), 127–132.
- [5] G. E. Andrews: Schur's theorem, Capparelli's conjecture, and the q -trinomial coefficients, *Contemp. Math.*, **166** (1994) 141–154.
- [6] G. E. Andrews and R. J. Baxter: Lattice gas generalization of the hard hexagon model III. q -trinomial coefficients, *J. Stat. Phys.*, **47** (1987), 297–330.
- [7] D. M. Bressoud: A combinatorial proof of Schur's 1926 partition theorem, *Proc. Amer. Math. Soc.*, **79** (1980), 338–340.
- [8] G. Gasper and M. Rahman: Basic hyper-geometric series, In: Encyclopedia of Mathematics and its Applications, Vol. 35, Cambridge, 1990.
- [9] W. Gleissberg: Über einen Satz von Herrn I. Schur, *Math. Z.*, **28** (1928), 372–382.
- [10] H. Göllnitz: Partitionen mit Differenzenbedingungen, *J. reine angew. Math.*, **225** (1967), 154–190.
- [11] I. J. Schur: Zur additiven Zahlentheorie *Gesammelte Abhandlungen*, Vol. 3, Springer (1973), 43–50.

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